

MODELING PHYSICAL SYSTEMS: HYDRODYNAMICS

PDE FINAL PROJECT

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1. INTRODUCTION

Throughout this semester, we have primarily focused on solving for and visualizing linear PDEs. Now, we aim to derive and interpret non-linear PDEs from a given physical system. In this paper, we will model shallow water waves over an uneven bottom. We will step through each physical law and assumption we make, analyzing the reasoning and limitations behind our modeling decisions. We will then linearize the equations that we derive, and analyze them to show qualitative and approximate behavior.

2. PHYSICAL SYSTEM AND GOVERNING EQUATIONS

Figure 1 illustrates the setup that we will be referencing and defines relevant variables. We will begin by claiming that the water is homogenous, inviscid, and irrotational.

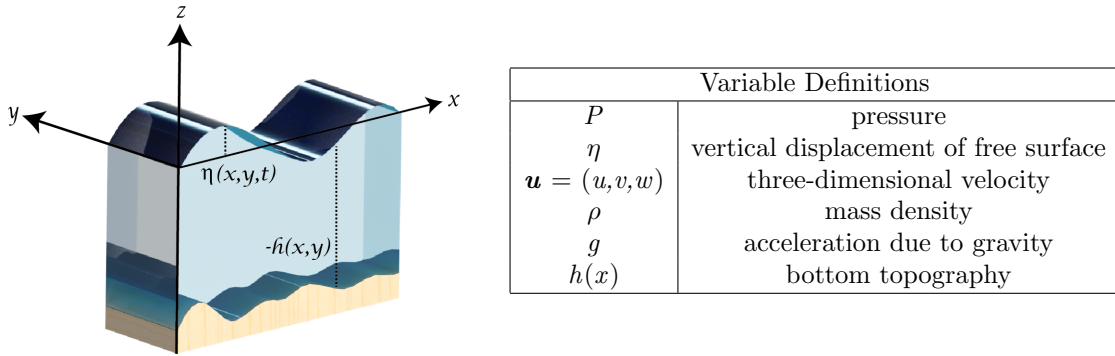


FIGURE 1. Illustration of Physical System

2.1. Conservation of Mass. We assume is that the system is incompressible/divergence-free, so we can apply the continuity equation.

$$\nabla \cdot \mathbf{u} = 0$$

We then expand the divergence of velocity in each direction and integrate with respect to height.

$$\begin{aligned} \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v + \frac{\partial}{\partial z} w &= 0 \\ \int \frac{\partial}{\partial x} u dz + \int \frac{\partial}{\partial y} v dz + \int \frac{\partial}{\partial z} w dz &= 0 \end{aligned}$$

In order to make sense of these integrals, recall Leibniz's rule:

$$\frac{\partial}{\partial \alpha} \int_a^b f(\alpha, z) dz = \int_a^b \frac{\partial}{\partial \alpha} f(\alpha, z) dz + \frac{\partial b}{\partial \alpha} f(\alpha, b) - \frac{\partial a}{\partial \alpha} f(\alpha, a)$$

For clarity, let's rearrange these terms and fill in the limits of integration. Note that α is a placeholder for direction (x, y, or z) and $f(\alpha, z)$ is the velocity in that direction.

$$\int_{-h}^{\eta} \frac{\partial}{\partial \alpha} f(\alpha, z) dz = \frac{\partial}{\partial \alpha} \int_{-h}^{\eta} f(\alpha, z) dz - \frac{\partial \eta}{\partial \alpha} f(\alpha, \eta) - \frac{\partial h}{\partial \alpha} f(\alpha, -h)$$

Using this expansion, we can rewrite our continuity equation as follows.

$$\frac{\partial}{\partial x} \int_{-h}^{\eta} u dz - \frac{\partial \eta}{\partial x} u|_{\eta} - \frac{\partial h}{\partial x} u|_{-h} +$$

$$\frac{\partial}{\partial y} \int_{-h}^{\eta} v dz - \frac{\partial \eta}{\partial y} v|_{\eta} - \frac{\partial h}{\partial y} v|_{-h} +$$

$$w|_{\eta} - w|_{-h} = 0$$

At this point, we want to introduce an important tool in hydrodynamic analysis: the material derivative. It describes the rate of change of a physical quantity of a parcel of water moving through a path in space over time. We can use this to capture the vertical displacement of water on its free surface (η) as a function of time and space.

$$\frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + \mathbf{u} \cdot \nabla \eta = w$$

$$\frac{\partial \eta}{\partial t} = w - \mathbf{u} \cdot \nabla \eta$$

$$\frac{\partial \eta}{\partial t} = w - \frac{\partial \eta}{\partial x} u - \frac{\partial \eta}{\partial y} v$$

We can use this relation to simplify our governing equation.

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \int_{-h}^{\eta} u dz + \frac{\partial}{\partial y} \int_{-h}^{\eta} v dz = 0$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} u(\eta + h) + \frac{\partial}{\partial y} v(\eta + h) = 0$$

$$\eta_t + u_x(\eta + h) + v_y(\eta + h) = 0$$

2.2. Conservation of Momentum. The next step in developing our model is to find a governing equation using Newton's Second Law.

$$\Sigma F = ma$$

$$\frac{D\mathbf{u}}{Dt} = \frac{\Sigma F}{m} = \frac{\nabla P}{\rho}$$

Note that the only force experienced by a water parcel is due to the pressure gradient. Pressure, in turn, is a function of mass density, gravity, and depth.

$$P = \int_z^{\eta} \rho g dz$$

We can equate these two expressions for force to get the conservation of momentum equation.

$$\frac{D\mathbf{u}}{Dt} + \frac{1}{\rho} \nabla P = 0$$

Since this material derivative is of the three-dimensional velocity vector, we can break this into three distinct equations.

X Direction:

$$\frac{Du}{Dt} = u_t + \begin{bmatrix} u \\ v \\ w \end{bmatrix} \cdot \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$$

$$\frac{Du}{Dt} + \frac{1}{\rho} P_x = u_t + uu_x + vu_y + wu_z + g(\eta - z)_x = 0$$

Y Direction:

$$\frac{Dv}{Dt} = v_t + \begin{bmatrix} u \\ v \\ w \end{bmatrix} \cdot \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

$$\frac{Dv}{Dt} + \frac{1}{\rho} P_y = v_t + uv_x + vv_y + wv_z + g(\eta - z)_y = 0$$

Z Direction:

$$\frac{Dw}{Dt} = w_t + \begin{bmatrix} u \\ v \\ w \end{bmatrix} \cdot \begin{bmatrix} w_x \\ w_y \\ w_z \end{bmatrix}$$

$$\frac{Dw}{Dt} + \frac{1}{\rho} P_z = w_t + uw_x + vw_y + ww_z + g(\eta - z)_z = 0$$

3. SHALLOW WATER EQUATIONS

What we have walked through thusfar is a generalized system for solving partial differential equations. We are able to derive the shallow wave PDEs that we seek by applying the long wave approximation. In such a system, the wavelength is considered to be much longer than the depth of the fluid and we can therefore neglect all terms relating to vertical changes in acceleration.

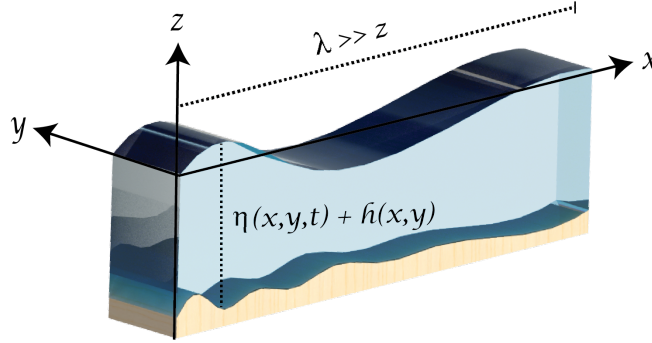


FIGURE 2. Shallow Wave System

The conservation of mass equation now simplifies to:

$$\eta_t + u_x(\eta + h) + v_y(\eta + h) = 0$$

$$\boxed{\eta_t + u_x(\eta + h) = 0}$$

Our three momentum equations now become two simpler equations only relating to x and y velocities.

$$u_t + uu_x + vv_y + wu_z + g(\eta - z)_x = 0$$

$$v_t + uv_x + vv_y + wv_z + g(\eta - z)_y = 0$$

$$\boxed{u_t + uu_x + vv_y + g\eta_x = 0}$$

$$\boxed{v_t + uv_x + vv_y + g\eta_y = 0}$$

4. ANALYSIS

4.1. Eigenvalue Analysis and Hyperbolic Functions. The shallow water equations are presented as a set of non-linear partial differential equations. There are many complex numerical methods that we can use to solve them, but we can also determine characteristics of the system through less rigorous means. For example, eigenvalue analysis may yield some interesting results. Let us rewrite our three equations in matrix notation.

$$\frac{\partial}{\partial t} \begin{pmatrix} \eta \\ u \\ v \end{pmatrix} + \begin{bmatrix} u & \eta + h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{bmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \eta \\ u \\ v \end{pmatrix} + \begin{bmatrix} v & 0 & \eta + h \\ 0 & v & 0 \\ g & 0 & v \end{bmatrix} \frac{\partial}{\partial y} \begin{pmatrix} \eta \\ u \\ v \end{pmatrix} = - \begin{pmatrix} u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \\ 0 \\ 0 \end{pmatrix}$$

For our first pass model, we assume that the waves are not changing in the y-direction such that $\frac{\partial}{\partial y} = 0$. While this assumption limits the accuracy of the model for real ocean-like systems, it allows us to better visually understand the behavior of the model. The same calculations could be carried out in the two-dimensional case. This simplifies our model to:

$$\frac{\partial}{\partial t} \begin{pmatrix} \eta \\ u \end{pmatrix} + \begin{bmatrix} u & \eta + h \\ g & u \end{bmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \eta \\ u \end{pmatrix} = - \begin{pmatrix} u \frac{\partial h}{\partial x} \\ 0 \end{pmatrix}$$

When a system is written in this matrix notation, the eigenvalues of the coefficient matrices are of particular interest. We move to analyze the below matrix and find eigenvalues $u \pm \sqrt{g(\eta + h)}$.

$$\begin{bmatrix} u & \eta + h \\ g & u \end{bmatrix}$$

These eigenvalues have units of velocity, and tell us about how to relate change in u and η . The change in the x-direction for slices in time is dictated by this relationship.

Since the eigenvalues are real and distinct, the shallow-water equations can be characterized as hyperbolic (wave-like) partial differential equations. For these functions, disturbances travel with a finite speed. Solutions to hyperbolic partial differential equations have several distinct features, the most prevalent being "transported" solutions. Some modifications are made to the exact boundaries of the waves, else we would be dealing with soliton waves, but it is this feature which differentiates them from parabolic (heat-like) solutions, which disperse in time. These equations further admit discontinuous solutions, which is to be expected: discontinuities (bores) approximate and are used to model breaking waves. The exact model for wave breaking is significantly more complex than the topics covered in this paper, and is worth considering as a future extension to this project.

4.2. Linearization. Nonlinear partial differential equations are difficult to analyze. Linearizing our system will allow us to derive wave speed, conduct vorticity analysis, and generally provide greater modelling opportunities. We will consider small disturbances about a fluid at rest in the area around (0,0). For linear approximations, we consider a neighboring point to be equal to the value at the current point added to the derivative. We will represent this mathematically for our two variables of interest:

$$\begin{aligned}\eta &= 0 + \eta' \\ u &= 0 + u'\end{aligned}$$

We can substitute this representation of small disturbance equations into our shallow wave equations, restated below.

$$\begin{aligned}\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \eta}{\partial x} &= 0 \\ \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[(\eta + h)u] &= 0\end{aligned}$$

For small disturbances, these equations become:

$$\begin{aligned}\frac{\partial u}{\partial t} + u' \frac{\partial u}{\partial x} + g \frac{\partial \eta}{\partial x} &= 0 \\ \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(\eta u) + \frac{\partial}{\partial x}(hu) &= 0\end{aligned}$$

The terms $u' \frac{\partial u}{\partial x}$ and $\frac{\partial}{\partial x}(\eta u)$ from the first and second equations, respectively, are second-order terms. We can neglect these for the time being, in the interest of creating first-order linear equations that we can model and extract characteristics from.

$$\begin{aligned}\frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} &= 0 \\ \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}(hu) &= 0\end{aligned}$$

Now, we can do some algebra to extract the linear 1D wave equation.

$$\begin{aligned}\sqrt{h} \left(\frac{\partial u}{\partial t} + g \frac{\partial \eta}{\partial x} \right) &= \frac{\partial}{\partial t}(u\sqrt{h}) + \sqrt{gh} \frac{\partial}{\partial x}(\eta\sqrt{g}) \\ \sqrt{g} \left(\frac{\partial \eta}{\partial t} + \frac{\partial hu}{\partial x} \right) &= \frac{\partial}{\partial t}(\eta\sqrt{g}) + \frac{\partial}{\partial x}(u\sqrt{h} \cdot g\sqrt{h})\end{aligned}$$

The purpose of rewriting the equations in this way is to allow the arguments of the partial differentials to take the same form. Now, if we eliminate the term $u\sqrt{h}$ from the equations, we obtain the linear 1D wave equation. We also take an extra step of redoing the $u = 0 + u'$ calculation.

$$\frac{\partial^2}{\partial t^2}(\eta\sqrt{g}) = \nabla \cdot [gh \cdot \nabla(\eta\sqrt{g})]$$

This also rewrites as:

$$(\eta\sqrt{g})_{tt} = (\sqrt{gh})^2 \nabla^2(\eta\sqrt{g})$$

In this form, we can see the obvious similarity to the standard wave equation, and observe that the wave speed is $c(x) = \sqrt{gh(x)}$, which shows that the wave speed explicitly depends on the varying boundary condition $h(x)$. Extending this to the two-dimensional case, we would find that the linearized wave equation takes the same form, with wave speed $c = \sqrt{gh(x,y)}$.

5. CONCLUSION

The shallow wave equations provide a good basis for understanding fluid boundary problems. There are many directions in which we could continue this project. Some revolve around further analysis of the equations themselves, including method of characteristics and vorticity analysis. Another asks us to consider the "breaking" of our model: breaking waves. In addition, we can use these equations and their linearized form to model physical phenomena.

For example, the linearized shallow water equations are a good model for the propagation of a tsunami across the open ocean while far from shore. We have shown above that the local speed of propagation of the tsunami, in any direction, is $c = \sqrt{gh(x)}$, with no dispersion. In linearizing the shallow-wave equations, we have removed any sort of dispersion relationship. For modeling a tsunami, the dispersion is less important due to the sheer size of the wave. When a tsunami is closer to the shore, the wave compresses horizontally and grows vertically, so the linear approximation is no longer valid.